Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

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Mathematics: analysis and approaches

Higher level

Paper 1

36 pages

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Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.

- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. $\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = \int\left(3-5x^{-\frac{1}{2}}\right)\mathrm{d}x$ ***(A1)***

$\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = 3x-10x^{\frac{1}{2}}\ (+c)$ ***A1A1***

substituting limits into their integrated function and subtracting ***(M1)***

$3(9)-10(9)^{\frac{1}{2}}-\left(3(1)-10(1)^{\frac{1}{2}}\right)$ OR $27-10\times 3-(3-10)$

$=4$ ***A1***

[5 marks]

2. (a) $\text{IQR} = 10 - 6\ (= 4)$ ***(A1)***

attempt to find $Q_3 + 1.5 \times \text{IQR}$ ***(M1)***

$10 + 6$

16 ***A1***

[3 marks]

(b) (i) choosing $c = \frac{1}{2}a - 9$ ***(M1)***

$\frac{1}{2} \times 42 - 9$

$= 12$ (years old) ***A1***

(ii) attempt to solve system by substitution or elimination ***(M1)***

34 (years old) ***A1***

[4 marks]

Total [7 marks]

3. (a) $(f \circ g)(x) = f(2x)$ ***(A1)***

$f(2x) = \sqrt{3}\sin 2x + \cos 2x$ ***A1***

[2 marks]

(b) $\sqrt{3}\sin 2x + \cos 2x = 2\cos 2x$

$\sqrt{3}\sin 2x = \cos 2x$

recognizing to use $\tan$ or $\cot$ ***M1***

$\tan 2x = \frac{1}{\sqrt{3}}$ OR $\cot 2x = \sqrt{3}$ (values may be seen in right triangle) ***(A1)***

$\left(\arctan\left(\frac{1}{\sqrt{3}}\right) =\right) \frac{\pi}{6}$ (seen anywhere) (accept degrees) ***(A1)***

$2x = \frac{\pi}{6}, \frac{7\pi}{6}$

$x = \frac{\pi}{12}, \frac{7\pi}{12}$ ***A1A1***

Note:	Do not award the final ***A1*** if any additional solutions are seen.
	Award ***A1A0*** for correct answers in degrees.
	Award ***A0A0*** for correct answers in degrees with additional values.

[5 marks]

Total [7 marks]

4. evidence of using product rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (2x-1)\times\left(k\mathrm{e}^{kx}\right) + 2\times\mathrm{e}^{kx} \quad \left(= \mathrm{e}^{kx}(2kx - k + 2)\right)$ ***A1***

correct working for one of (seen anywhere) ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1 \Rightarrow k\mathrm{e}^{k} + 2\mathrm{e}^{k}$

OR

slope of tangent is $5\mathrm{e}^{k}$

their $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1$ equals the slope of $y = 5\mathrm{e}^{k}x \left(= 5\mathrm{e}^{k}\right)$ (seen anywhere) ***(M1)***

$k\mathrm{e}^{k} + 2\mathrm{e}^{k} = 5\mathrm{e}^{k}$

$k = 3$ ***A1***

[5 marks]

5. (a) translation (shift) by $\frac{3\pi}{2}$ to the right OR positive horizontal direction by $\frac{3\pi}{2}$ ***A1***

translation (shift) by q upwards OR positive vertical direction by q ***A1***

Note: accept translation by $\begin{pmatrix} \frac{3\pi}{2} \\ q \end{pmatrix}$

Do not accept 'move' for translation/shift.

[2 marks]

(b) **METHOD 1**

minimum of $4\sin\left(x-\frac{3\pi}{2}\right)$ is -4 (may be seen in sketch) ***(M1)***

$-4+2.5+q \geq 7$

$q \geq 8.5$ (accept $q=8.5$) ***A1***

substituting $x=0$ and their $q(=8.5)$ to find r ***(M1)***

$(r=)\quad 4\sin\left(\frac{-3\pi}{2}\right)+2.5+8.5$

$4+2.5+8.5$ ***(A1)***

smallest value of r is 15 ***A1***

continued…

Question 5 continued

METHOD 2

substituting $x=0$ to find an expression (for r) in terms of q ***(M1)***

$\left(g(0)=r=\right)\quad 4\sin\left(\frac{-3\pi}{2}\right)+2.5+q$

$\left(r=\right)\ 6.5+q$ ***A1***

minimum of $4\sin\left(x-\frac{3\pi}{2}\right)$ is -4 ***(M1)***

$-4+2.5+q\geq 7$

$-4+2.5+(r-6.5)\geq 7$ (accept =) ***(A1)***

smallest value of r is 15 ***A1***

METHOD 3

$4\sin\left(x-\frac{3\pi}{2}\right)+2.5+q=4\cos x+2.5+q$ ***A1***

y-intercept of $4\cos x+2.5+q$ is a maximum ***(M1)***

amplitude of $g(x)$ is 4 ***(A1)***

attempt to find least maximum ***(M1)***

$r=2\times 4+7$

smallest value of r is 15 ***A1***

[5 marks]

Total [7 marks]

6. **EITHER**

attempt to obtain the general term of the expansion

$T_{r+1} = {}^{n}C_{r}\left(8x^{3}\right)^{n-r}\left(-\frac{1}{2x}\right)^{r}$ OR $T_{r+1} = {}^{n}C_{n-r}\left(8x^{3}\right)^{r}\left(-\frac{1}{2x}\right)^{n-r}$ ***(M1)***

OR

recognize power of x starts at $3n$ and goes down by 4 each time ***(M1)***

THEN

recognizing the constant term when the power of x is zero (or equivalent) ***(M1)***

$r = \frac{3n}{4}$ or $n = \frac{4}{3}r$ or $3n - 4r = 0$ OR $3r - (n - r) = 0$ (or equivalent) ***A1***

r is a multiple of 3 $(r = 3, 6, 9, \ldots)$ or one correct value of n (seen anywhere) ***(A1)***

$n = 4k, k \in \mathbb{Z}^{+}$ ***A1***

Note:	Accept n is a (positive) multiple of 4 or $n = 4, 8, 12, \ldots$
	Do not accept $n = 4, 8, 12$
Note:	Award full marks for a correct answer using trial and error approach showing $n = 4, 8, 12, \ldots$ and for recognizing that this pattern continues.

[5 marks]

7. (a) attempt to integrate $\frac{k}{\sqrt{4-3x^2}}$ ***(M1)***

$$= k\left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]$$ ***A1***

Note: Award ***(M1)A0*** for $\arcsin\left(\frac{\sqrt{3}}{2}x\right)$. Condone absence of k up to this stage.

equating their integrand to 1 ***M1***

$$k\left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]_0^1 = 1$$

$$k = \frac{3\sqrt{3}}{\pi}$$ ***A1***

[4 marks]

continued…

Question 7 continued

(b) $\mathrm{E}(X)=\frac{3\sqrt{3}}{\pi}\int_0^1 \frac{x}{\sqrt{4-3x^2}}\mathrm{d}x$ ***A1***

Note: Condone absence of limits if seen at a later stage.

EITHER

attempt to integrate by inspection ***(M1)***

$$=\frac{3\sqrt{3}}{\pi}\times-\frac{1}{6}\int -6x\left(4-3x^2\right)^{-\frac{1}{2}}\mathrm{d}x$$

$$=\frac{3\sqrt{3}}{\pi}\left[-\frac{1}{3}\sqrt{4-3x^2}\right]_0^1$$ ***A1***

Note: Condone the use of k up to this stage.

OR

for example, $u=4-3x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=-6x$

Note: Other substitutions may be used. For example, $u=-3x^2$.

$$=-\frac{\sqrt{3}}{2\pi}\int_4^1 u^{-\frac{1}{2}}\mathrm{d}u$$ ***M1***

Note: Condone absence of limits up to this stage.

$$=-\frac{\sqrt{3}}{2\pi}\left[2\sqrt{u}\right]_4^1$$ ***A1***

Note: Condone the use of k up to this stage.

THEN

$$=\frac{\sqrt{3}}{\pi}$$ ***A1***

Note: Award ***A0M1A1A0*** for their $k\left[-\frac{1}{3}\sqrt{4-3x^2}\right]$ or $k\left[-2\sqrt{u}\right]$ for working with incorrect or no limits.

[4 marks]

Total [8 marks]

8. Assume that a and b are both odd. ***M1***

Note:	Award ***M0*** for statements such as "let a and b be both odd".
Note:	Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

Then $a = 2m+1$ and $b = 2n+1$ ***A1***

$a^2 + b^2 \equiv (2m+1)^2 + (2n+1)^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$ ***A1***

$= 4\left(m^2 + m + n^2 + n\right) + 2$ ***(A1)***

($4\left(m^2 + m + n^2 + n\right)$ is always divisible by 4) but 2 is not divisible by 4. (or equivalent) ***R1***

$\Rightarrow a^2 + b^2$ is not divisible by 4, a contradiction. (or equivalent) ***R1***

hence a and b cannot both be odd. ***AG***

Note:	Award a maximum of ***M1A0A0(A0)R1R1*** for considering identical or two consecutive odd numbers for a and b.

[6 marks]

9. (a) $z_1z_2 = (1+b\mathrm{i})\left((1-b^2)-(2b)\mathrm{i}\right)$

$= (1-b^2-2\mathrm{i}^2b^2)+\mathrm{i}(-2b+b-b^3)$ ***M1***

$= (1+b^2)+\mathrm{i}(-b-b^3)$ ***A1A1***

Note: Award ***A1*** for $1+b^2$ and ***A1*** for $-b\mathrm{i}-b^3\mathrm{i}$.

[3 marks]

(b) $\arg(z_1z_2) = \arctan\left(\dfrac{-b-b^3}{1+b^2}\right) = \dfrac{\pi}{4}$ ***(M1)***

EITHER

$\arctan(-b) = \dfrac{\pi}{4}$ (since $1+b^2 \neq 0, \text{for } b \in \mathbb{R}$) ***A1***

OR

$-b-b^3 = 1+b^2$ (or equivalent) ***A1***

THEN

$b = -1$ ***A1***

[3 marks]

Total [6 marks]

Section B

10. (a) (i) **EITHER**

attempt to use a ratio from consecutive terms ***M1***

$$\frac{p\ln x}{\ln x}=\frac{\frac{1}{3}\ln x}{p\ln x} \quad \text{OR} \quad \frac{1}{3}\ln x=(\ln x)r^2 \quad \text{OR} \quad p\ln x=\ln x\left(\frac{1}{3p}\right)$$

Note: Candidates may use $\ln x^1+\ln x^p+\ln x^{\frac{1}{3}}+\ldots$ and consider the powers of x in geometric sequence. Award ***M1*** for $\frac{p}{1}=\frac{\frac{1}{3}}{p}$.

OR

$r=p$ and $r^2=\frac{1}{3}$ ***M1***

THEN

$p^2=\frac{1}{3}$ OR $r=\pm\frac{1}{\sqrt{3}}$ ***A1***

$p=\pm\frac{1}{\sqrt{3}}$ ***AG***

Note: Award ***M0A0*** for $r^2=\frac{1}{3}$ or $p^2=\frac{1}{3}$ with no other working seen.

continued…

Question 10 continued

(ii) **EITHER**

since, $|p| = \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}} < 1$ ***R1***

OR

since, $|p| = \frac{1}{\sqrt{3}}$ and $-1 < p < 1$ ***R1***

THEN

$\Rightarrow$ the geometric series converges. ***AG***

Note: Accept r instead of p.
Award ***R0*** if both values of p not considered.

(iii) $\frac{\ln x}{1 - \frac{1}{\sqrt{3}}} \quad \left(= 3 + \sqrt{3}\right)$ ***(A1)***

$\ln x = 3 - \frac{3}{\sqrt{3}} + \sqrt{3} - \frac{\sqrt{3}}{\sqrt{3}}$ OR $\ln x = 3 - \sqrt{3} + \sqrt{3} - 1 \quad (\Rightarrow \ln x = 2)$ ***A1***

$x = \mathrm{e}^2$ ***A1***

[6 marks]

continued…

Question 10 continued

(b) (i) **METHOD 1**

attempt to find a difference from consecutive terms or from u_2 ***M1***

correct equation ***A1***

$p\ln x-\ln x=\frac{1}{3}\ln x-p\ln x$ OR $\frac{1}{3}\ln x=\ln x+2(p\ln x-\ln x)$

Note: Candidates may use $\ln x^1+\ln x^p+\ln x^{\frac{1}{3}}+\ldots$ and consider the powers of x in arithmetic sequence. Award ***M1A1*** for $p-1=\frac{1}{3}-p$.

$2p\ln x=\frac{4}{3}\ln x\ \left(\Rightarrow 2p=\frac{4}{3}\right)$ ***A1***

$p=\frac{2}{3}$ ***AG***

METHOD 2

attempt to use arithmetic mean $u_2=\frac{u_1+u_3}{2}$ ***M1***

$p\ln x=\frac{\ln x+\frac{1}{3}\ln x}{2}$ ***A1***

$2p\ln x=\frac{4}{3}\ln x\ \left(\Rightarrow 2p=\frac{4}{3}\right)$ ***A1***

$p=\frac{2}{3}$ ***AG***

continued…

Question 10 continued

METHOD 3

attempt to find difference using u_3 ***M1***

$\frac{1}{3}\ln x = \ln x + 2d \quad \left(\Rightarrow d = -\frac{1}{3}\ln x\right)$

$u_2 = \ln x + \frac{1}{2}\left(\frac{1}{3}\ln x - \ln x\right)$ OR $p\ln x - \ln x = -\frac{1}{3}\ln x$ ***A1***

$p\ln x = \frac{2}{3}\ln x$ ***A1***

$p = \frac{2}{3}$ ***AG***

(ii) $d = -\frac{1}{3}\ln x$ ***A1***

continued…

Question 10 continued

(iii) **METHOD 1**

$$S_n = \frac{n}{2}\left[2\ln x + (n-1)\times\left(-\frac{1}{3}\ln x\right)\right]$$

attempt to substitute into S_n and equate to $\ln\left(\frac{1}{x^3}\right)$ ***(M1)***

$$\frac{n}{2}\left[2\ln x + (n-1)\times\left(-\frac{1}{3}\ln x\right)\right] = \ln\left(\frac{1}{x^3}\right)$$

$\ln\left(\frac{1}{x^3}\right) = -\ln x^3 \left(= \ln x^{-3}\right)$ ***(A1)***

$= -3\ln x$ ***(A1)***

correct working with S_n (seen anywhere) ***(A1)***

$\frac{n}{2}\left[2\ln x - \frac{n}{3}\ln x + \frac{1}{3}\ln x\right]$ OR $n\ln x - \frac{n(n-1)}{6}\ln x$ OR $\frac{n}{2}\left(\ln x + \left(\frac{4-n}{3}\right)\ln x\right)$

correct equation without $\ln x$ ***A1***

$\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$ OR $n - \frac{n(n-1)}{6} = -3$ (or equivalent)

Note: Award as above if the series $1 + p + \frac{1}{3} + \dots$ is considered leading to $\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$.

attempt to form a quadratic $= 0$ ***(M1)***

$n^2 - 7n - 18 = 0$

attempt to solve their quadratic ***(M1)***

$(n-9)(n+2) = 0$

$n = 9$ ***A1***

continued…

Question 10 continued

METHOD 2

$\ln\left(\frac{1}{x^3}\right) = -\ln x^3 \left(= \ln x^{-3}\right)$ ***(A1)***

$= -3\ln x$ ***(A1)***

listing the first 7 terms of the sequence ***(A1)***

$\ln x + \frac{2}{3}\ln x + \frac{1}{3}\ln x + 0 - \frac{1}{3}\ln x - \frac{2}{3}\ln x - \ln x + \ldots$

recognizing first 7 terms sum to 0 ***M1***

8^{th} term is $-\frac{4}{3}\ln x$ ***(A1)***

9^{th} term is $-\frac{5}{3}\ln x$ ***(A1)***

sum of 8^{th} and 9^{th} terms $= -3\ln x$ ***(A1)***

$n = 9$ ***A1***

[12 marks]

Total [18 marks]

11. (a) **METHOD 1**

attempt to eliminate a variable *M1*

obtain a pair of equations in two variables

EITHER

$-3x + z = -3$ and *A1*

$-3x + z = 44$ *A1*

OR

$-5x + y = -7$ and *A1*

$-5x + y = 40$ *A1*

OR

$3x - z = 3$ and *A1*

$3x - z = -\frac{79}{5}$ *A1*

THEN

the two lines are parallel ($-3 \neq 44 \text{ or } -7 \neq 40 \text{ or } 3 \neq -\frac{79}{5}$) *R1*

Note:	There are other possible pairs of equations in two variables. To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect *AG*

[4 marks]

continued…

Question 11 continued

METHOD 2

vector product of the two normals $= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$ (or equivalent) ***A1***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if "$r =$" is missing. Subsequent marks may still be awarded.

Attempt to substitute $(1+\lambda, -2+5\lambda, 3\lambda)$ in Π_3 ***M1***

$-9(1+\lambda)+3(-2+5\lambda)-2(3\lambda)=32$

$-15=32,$ a contradiction ***R1***

hence the three planes do not intersect ***AG***

[4 marks]

continued…

Question 11 continued

METHOD 3

attempt to eliminate a variable ***M1***

$-3y+5z=6$ ***A1***

$-3y+5z=100$ ***A1***

$0=94$, a contradiction ***R1***

Note:	Accept other equivalent alternatives. Accept other valid methods. To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

[4 marks]

continued…

Question 11 continued

(b) (i) $\Pi_1 : 2+2+0=4$ and $\Pi_2 : 1+4+0=5$ ***A1***

(ii) **METHOD 1**

attempt to find the vector product of the two normals ***M1***

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$$ ***A1***

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ ***A1A1***

Note: Award ***A1A0*** if "$r =$" is missing.
Accept any multiple of the direction vector.

Working for (b)(ii) may be seen in part (a) Method 2. In this case penalize lack of "$r =$" only once.

continued…

Question 11 continued

METHOD 2

attempt to elimnate a variable from Π_1 and Π_2 ***M1***

$3x - z = 3$ OR $3y - 5z = -6$ OR $5x - y = 7$

Let $x = t$

substituting $x = t$ in $3x - z = 3$ to obtain

$z = -3 + 3t$ and $y = 5t - 7$ (for all three variables in parametric form) ***A1***

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ ***A1A1***

Note:	Award ***A1A0*** if "$r =$" is missing. Accept any multiple of the direction vector. Accept other position vectors which satisfy both the planes Π_1 and Π_2.

[5 marks]

continued…

Question 11 continued

(c) **METHOD 1**

the line connecting L and Π_3 is given by L_1

attempt to substitute position and direction vector to form L_1 ***(M1)***

$$\boldsymbol{s} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix}$$ ***A1***

substitute $(1-9t, -2+3t, -2t)$ in Π_3 ***M1***

$-9(1-9t)+3(-2+3t)-2(-2t)=32$

$94t = 47 \Rightarrow t = \frac{1}{2}$ ***A1***

attempt to find distance between $(1,-2,0)$ and their point $\left(-\frac{7}{2}, -\frac{1}{2}, -1\right)$ ***(M1)***

$$= \left|\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right| = \frac{1}{2}\sqrt{(-9)^2 + 3^2 + (-2)^2}$$

$$= \frac{\sqrt{94}}{2}$$ ***A1***

[6 marks]

continued…

Question 11 continued

METHOD 2

unit normal vector equation of Π_3 is given by $\dfrac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}}$ ***(M1)***

$= \dfrac{32}{\sqrt{94}}$ ***A1***

let Π_4 be the plane parallel to Π_3 and passing through P,

then the normal vector equation of Π_4 is given by

$\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix} = -15$ ***M1***

unit normal vector equation of Π_4 is given by

$\dfrac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}} = \dfrac{-15}{\sqrt{94}}$ ***A1***

distance between the planes is $\dfrac{32}{\sqrt{94}} - \dfrac{-15}{\sqrt{94}}$ ***(M1)***

$= \dfrac{47}{\sqrt{94}} \left(= \dfrac{\sqrt{94}}{2}\right)$ ***A1***

[6 marks]

Total [15 marks]

12. (a) **METHOD 1**

recognition of both known series ***(M1)***

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots \text{ and } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

attempt to multiply the two series up to and including x^3 term ***(M1)***

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$$= x - \frac{x^3}{3!} + x^2 + \frac{x^3}{2!} + \ldots$$ ***(A1)***

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$$ ***A1***

[4 marks]

METHOD 2

$$f(x) = e^x \sin x$$

$$f'(x) = e^x \cos x + e^x \sin x$$ ***A1***

$$f''(x) = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x \left(= 2e^x \cos x\right)$$

$$f'''(x) = 2e^x \cos x - 2e^x \sin x$$

$f''(x) = 2e^x \cos x$ and $f'''(x) = 2e^x\left(\cos x - \sin x\right)$ ***A1***

substitute $x = 0$ into f or its derivatives to obtain Maclaurin series ***(M1)***

$$e^x \sin x = 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times 2 + \ldots$$

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$$ ***A1***

[4 marks]

continued…

Question 12 continued

(b) $\mathrm{e}^{x^2}\sin\left(x^2\right)=x^2+x^4+\frac{1}{3}x^6+\ldots$ ***(A1)***

substituting their expression and attempt to integrate ***M1***

$$\int_0^1 \mathrm{e}^{x^2}\sin(x^2)\,\mathrm{d}x \approx \int_0^1\left(x^2+x^4+\frac{1}{3}x^6\right)\mathrm{d}x$$

Note: Condone absence of limits up to this stage.

$$=\left[\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{21}\right]_0^1$$ ***A1***

$$=\frac{61}{105}$$ ***A1***

[4 marks]

continued…

Question 12 continued

(c) (i) attempt to use product rule at least once **M1**

$g'(x) = e^x \cos x - e^x \sin x$ **A1**

$g''(x) = e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x \left(= -2e^x \sin x\right)$ **A1**

EITHER

$2\left(g'(x) - g(x)\right) = 2\left(e^x \cos x - e^x \sin x - e^x \cos x\right) = -2e^x \sin x$ **A1**

OR

$g''(x) = 2\left(e^x \cos x - e^x \sin x - e^x \cos x\right)$ **A1**

THEN

$g''(x) = 2\left(g'(x) - g(x)\right)$ **AG**

Note: Accept working with each side separately to obtain $-2e^x \sin x$.

(ii) $g'''(x) = 2\left(g''(x) - g'(x)\right)$ **A1**

$g^{(4)}(x) = 2\left(g'''(x) - g''(x)\right)$ **AG**

Note: Accept working with each side separately to obtain $-4e^x \cos x$.

[5 marks]

continued…

Question 12 continued

(d) attempt to substitute $x = 0$ into a derivative **(M1)**

$g(0) = 1,\ g'(0) = 1,\ g''(0) = 0$ **A1**

$g'''(0) = -2,\ g^{(4)}(0) = -4$ **(A1)**

attempt to substitute into Maclaurin formula **(M1)**

$g(x) = 1 + x - \frac{2}{3!}x^3 - \frac{4}{4!}x^4 + \ldots\left(= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \ldots\right)$ **A1**

Note: Do not award any marks for approaches that do not use the part (c) result.

[5 marks]

(e) **METHOD 1**

$\lim_{x\to 0}\frac{e^x\cos x - 1 - x}{x^3} = \lim_{x\to 0}\frac{\left(1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + \ldots\right) - 1 - x}{x^3}$ **M1**

$= \lim_{x\to 0}\left(-\frac{1}{3} - \frac{1}{6}x + \ldots\right)$ **(A1)**

$= -\frac{1}{3}$ **A1**

Note: Condone the omission of $+\ldots$ in their working.

continued…

Question 12 continued

METHOD 2

$\lim_{x\to 0}\frac{e^x\cos x-1-x}{x^3}=\frac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule ***M1***

$=\lim_{x\to 0}\frac{e^x\cos x-e^x\sin x-1}{3x^2}\left(=\lim_{x\to 0}\frac{g'(x)-1}{3x^2}\right)$

$=\frac{0}{0}$, using l'Hôpital's rule again

$=\lim_{x\to 0}\frac{-2e^x\sin x}{6x}\left(=\lim_{x\to 0}\frac{g''(x)}{6x}\right)$

$=\frac{0}{0}$, using l'Hôpital's rule again

$=\lim_{x\to 0}\frac{-2e^x\sin x-2e^x\cos x}{6}\left(=\lim_{x\to 0}\frac{g'''(x)}{6}\right)$ ***A1***

$=-\frac{1}{3}$ ***A1***

[3 marks]

Total [21 marks]